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SURFACES OF ROTATION IN A SPACE OF FOUR DIMENSIONS.

BY C. L. E. MOORE.

1. The obvious generalization of a surface of rotation in ordinary space is a surface left invariant by a rotation in four dimensions, a rotation being defined as a linear transformation of positive determinant preserving distance and leaving one point fixed. In general the path curve of a point by a rotation in four dimensions is not a circle.[*] There are however two distinct types of rotations for which these path curves are circles. A rotation in general leaves two completely perpendicular planes invariant, not fixed point for point, but only as planes.[†] A special rotation leaves one of these planes fixed point for point. For this rotation the path curves are circles whose centers are in the second fixed plane and whose planes are parallel to the absolutely fixed plane. Surfaces generated by this kind of rotation have been studied somewhat.[‡] If the rate of rotation in each of the invariant planes is the same the path curves are circles having their centers at the origin.[§] In this note I shall confine my attention mostly to surfaces left invariant by this last kind of rotation. The path curves now however do not lie in parallel planes.

A general rotation in a space of four dimensions can be written

$$(1)\qquad \begin{aligned} X_1 &= x_1 \cos m_1 t - x_2 \sin m_1 t, & X_2 &= x_1 \sin m_1 t + x_2 \cos m_1 t, \\ X_3 &= x_3 \cos m_2 t - x_4 \sin m_2 t, & X_4 &= x_3 \sin m_2 t + x_4 \cos m_2 t, \end{aligned}$$

where the invariant planes are the X_1X_2- and the X_3X_4-planes, and m_1, m_2 are the rates of rotation in these planes. If the X_3X_4-plane is absolutely fixed, $m_2 = 0$. Surfaces generated by this rotation are the ones discussed by Wilson and Moore. The rotations we shall study now are given by $m_1 = \pm m_2$, but for the sake of simplicity we shall confine our attention to the case $m_1 = m_2 = 1$. The equations of the rotation then become

$$(2)\qquad \begin{aligned} X_1 &= x_1 \cos t - x_2 \sin t, & X_3 &= x_3 \cos t - x_4 \sin t, \\ X_2 &= x_1 \sin t + x_2 \cos t, & X_4 &= x_3 \sin t + x_4 \cos t. \end{aligned}$$

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[*] See C. L. E. Moore, Motions in hyperspace, Annals of Math., second series, vol. 19 (1918), pp. 176–184.

[†] Cole, On rotations in four dimensions, American Journal of Mathematics, vol. 12 (1889), pp. 191–210.

[‡] Wilson and Moore, Differential geometry of two-dimensional surfaces in hyperspace, Proceedings of the American Academy of Arts and Sciences, vol. 52 (1916), pp. 267–368.

[§] Moore, Rotations in hyperspace, Proceedings of the American Academy of Arts and Sciences, vol. 53 (1918), pp. 649–694.

From (2) it is at once evident that

$$\sum_1^4 X_i^2 = \sum_1^4 x_i^2.$$

Consequently distance from the origin is left unchanged. Also if we allow t to vary, the curve which the point (X_1, X_2, X_3, X_4) describes will lie in the plane

$$\begin{aligned} AX_1 + BX_2 + CX_3 + DX_4 &= 0, \\ BX_1 - AX_2 + DX_3 - CX_4 &= 0, \end{aligned} \tag{3}$$

where A, B, C, D are so chosen that

$$\begin{aligned} Ax_1 + Bx_2 + Cx_3 + Dx_4 &= 0, \\ Bx_1 - Ax_2 + Dx_4 - Cx_3 &= 0. \end{aligned}$$

Consequently if (x_1, x_2, x_3, x_4) lies in the plane (3), the curve (2) must lie in the same plane. Hence *any plane* (3) *is left invariant by the rotation* (2), *and consequently the path curve of a point lying in that plane must be a circle with center at the origin.*

The plane (3) cuts the planes

$$\begin{cases} X_1 + iX_2 = 0, \\ X_3 + iX_4 = 0, \end{cases} \qquad \begin{cases} X_1 - iX_2 = 0, \\ X_3 - iX_4 = 0, \end{cases} \tag{4}$$

in lines for every value of A, B, C, D. The planes left invariant by the rotation form the linear congruence of planes cutting the planes (4). This congruence will contain the plane completely perpendicular to any plane (3) as can easily be verified. A linear congruence of planes, in space of four dimensions, which pass through a fixed point is simply isomorphic with a linear congruence of lines in ordinary space. We see then that one and only one plane of (3) contains any line which passes through the origin unless the given line lies in one of the planes (4).

2. **Rotation of a plane.** If a plane π passes through O, it will cut ∞^1 planes of (3) in lines. These planes then cut three fixed planes in lines and hence form one system of generators of a quadric hypercone which has O for simple vertex. The different positions which π takes will form the second system of generators of the cone. If the plane does not pass through O, let its equations be

$$\sum_1^4 A_iX_i + A_5 = 0, \qquad \sum_1^4 B_iX_i + B_5 = 0. \tag{5}$$

If we apply the transformation (2) to this plane, we have on eliminating t

$$\begin{vmatrix} A_5 & \lambda_1 \\ B_5 & \lambda_2 \end{vmatrix}^2 + \begin{vmatrix} \mu_1 & A_5 \\ \mu_2 & B_5 \end{vmatrix}^2 = \begin{vmatrix} \lambda_1 & \mu_1 \\ \lambda_2 & \mu_2 \end{vmatrix}^2, \tag{6}$$

where

$$\lambda_1 = \sum_1^4 A_i x_i, \qquad \lambda_2 = \sum_1^4 B_i x_i,$$
$$\mu_1 = A_2x_1 - A_1x_2 + A_4x_3 - A_3x_4,$$
$$\mu_2 = B_2x_1 - B_1x_2 + B_4x_3 - B_3x_4.$$

Hence *the hypersurface generated by revolving a plane not passing through the origin is of order four and has a double point at the origin.* If the plane passes through O, $A_5 = B_5 = 0$ and (6) reduces to a quadric cone. If λ_1, λ_2, μ_1, μ_2 have a line in common, the cone will have this common line for vertex. But we see that $\lambda_1 = 0$, $\mu_1 = 0$ forms a plane of (3), and likewise $\lambda_2 = 0$, $\mu_2 = 0$. Two planes (3) can only intersect in a line lying in one of the planes (4). Hence if a plane cutting one of the planes (4) in a line is rotated, it will generate a cone having this line for vertex. These cones are evidently imaginary.

The general hypersurface of order four above will cut each of the invariant planes in a circle, since two planes in space of four dimensions always have a point in common and the point will rotate in a circle. However, if the plane passes through O, the quadric cone generated will cut only a simple infinity of the invariant planes in circles but will cut each of these in an infinite number of concentric circles.

Two near by positions of the plane (5) will have a point in common. The locus of this point will be a circle to which all the planes are tangent. For the plane near to (5) is

$$(7)\quad \begin{cases} A_1(x_1 - x_2dt) + A_2(x_1dt + x_2) + A_3(x_3 - x_4dt) + A_4(x_3dt + x_4) = 0, \\ B_1(x_1 - x_2dt) + B_2(x_1dt + x_2) + B_3(x_3 - x_4dt) + B_4(x_3dt + x_4) = 0. \end{cases}$$

By means of (5) equations (7) are reducible to

$$(8)\qquad \begin{cases} A_2x_1 - A_1x_2 + A_4x_3 - A_3x_4 = 0, \\ B_2x_1 - B_1x_2 + B_4x_3 - B_3x_4 = 0. \end{cases}$$

In equations (2) we consider x_i as fixed; this will be the point corresponding to $t = 0$. The point corresponding to the value $t = dt$ will be

$$(x_1 - x_2dt,\ x_1dt + x_2,\ x_3 - x_4dt,\ x_3dt + x_4),$$

and equation (7) is the condition that this point lie in (5). Consequently the plane (5) will be tangent to the locus of intersection of (5) and (8). If (5) and (8) have more than one point in common, they will evidently have a straight line in common. Hence *if a plane is rotated it will always be tangent to one of the invariant circles (path curves), and if it is tangent to more than one of them it will be tangent to an infinite number and the points of tangency will all lie on a straight line.*

If (5) and (8) have a line in common, the hypercone (6) will have this line for vertex.

3. **Rotation of a line.** If a line passing through O is rotated, it will generate the invariant plane containing this line, or if the line lies in an invariant plane and does not pass through O it will generate this plane when rotated. If the line does not pass through O nor lie in an invariant plane, two planes can be passed through it, one containing O and the other not. Then the surface formed by rotating the line will be the intersection of the hypersurfaces formed by rotating these two planes. Hence *the surface formed by rotating a line is of order eight and will have a multiple point at the origin.* The only developable surfaces (that is generated by the tangent lines to a curve) are planes.

If in (2) we make x_i functions of s (arc length), equations (2) become the parametric equations of the surface formed by rotating the curve

$$x_i = x_i(s) \qquad (i = 1, 2, 3, 4). \tag{9}$$

The vector equation of the surface is

$$\rho = \Sigma X_i k_i. \tag{10}$$

Then

$$\left\{\begin{aligned} m &= \frac{\partial \rho}{\partial s} = (x_1' \cos t - x_2' \sin t)k_1 + (x_1' \sin t + x_2' \cos t)k_2 \\ &\qquad + (x_3' \cos t - x_4' \sin t)k_3 + (x_3' \sin t + x_4' \cos t)k_4, \\ n &= \frac{\partial \rho}{\partial t} = -(x_1 \sin t + x_2 \cos t)k_1 + (x_1 \cos t - x_2 \sin t)k_2 \\ &\qquad - (x_3 \sin t + x_4 \cos t)k_3 + (x_3 \cos t - x_4 \sin t)k_4, \end{aligned}\right. \tag{11}$$

where primes indicate derivatives with respect to s.

Now

$$m \cdot n = x_1'x_2 - x_2'x_1 + x_3'x_4 - x_4'x_3. \tag{12}$$

Hence the condition that the curve (9) be orthogonal to the path curves cutting it is

$$x_1'x_2 - x_2'x_1 + x_3'x_4 - x_4'x_3 = 0. \tag{13}$$

If (9) is a straight line, x_i will be linear functions of s

$$x_i = a_i s + b_i,$$

and we obtain on substituting in (13)

$$a_1b_2 - a_2b_1 + a_3b_4 - a_4b_3 = 0, \tag{14}$$

as the condition that the rulings on a ruled surface of rotation cut the path curves orthogonally. Since (14) does not contain the parameter s, we see that if one path curve cuts a straight line at right angles, every path curve cutting it will cut it at right angles. Hence passing through each point of space are ∞^2 lines which are orthogonal to every path curve they cut, viz., the lines passing through the point and lying in the space of three dimensions perpendicular to the path curve through the point. In each plane there is a pencil of such lines. Also if two intersecting lines are orthogonal to the path curve through the point of intersection, each line of the pencil formed by them will have the same property.

4. **Rotation of plane curves.** If (9) defines a curve lying in the plane

$$(15)\qquad \begin{aligned} x_1 &= A_1x_2 + B_1x_4 + C_1, \\ x_3 &= A_2x_2 + B_2x_4 + C_2, \end{aligned}$$

then the condition that the curve be orthogonal to the path curves is

$$(16)\qquad (B_1 - A_2)(x_2x_4' - x_2'x_4) - C_1x_2' - C_2x_4' = 0.$$

The solution of this equation is

$$(B_1 - A_2)x_4 - C_1 = k[(B_1 - A_2)x_2 - C_2].$$

This equation together with (15) determines a pencil of lines in (15) which are the only curves lying in a general plane which cut the path curves orthogonally. However if (16) is identically satisfied $B_1 = A_2$, $C_1 = C_2 = 0$, and (15) becomes

$$(17)\qquad \begin{aligned} x_1 &= A_1x_2 + B_1x_4, \\ x_3 &= B_1x_2 + B_2x_4. \end{aligned}$$

Then every path curve cutting (17) cuts it orthogonally. It is easily seen that the plane passing through O and completely perpendicular to (3) is

$$(18)\qquad \begin{aligned} (AC + BD)x_1 + (BC - AD)x_2 - (A^2 + B^2)x_3 &= 0, \\ (AD - BC)x_1 + (AC + BD)x_2 - (A^2 + B^2)x_4 &= 0. \end{aligned}$$

This plane is also seen to belong to the system of invariant planes. If the plane (17) cuts (3) in a line, it is easily shown that it will also cut (18) in a line. A general plane cutting (3) and (18) in lines must have two essential constants. Equation (17) has two essential constants and consequently is the general form of the equation of a plane which if it cuts one invariant plane in a line through the origin will also cut the invariant plane completely perpendicular to it in a line. That is the invariant planes cutting (17) can be grouped into completely perpendicular pairs. Hence *if a curve lies in a plane which cuts two completely perpendicular planes, it will be orthogonal to every path curve which cuts it.*

5. **Some particular surfaces.** The element of arc of the surface (10) is

$$d\sigma^2 = m\cdot m ds^2 + 2m\cdot n ds dt + n\cdot n dt^2.$$

If the generating curve (9) is orthogonal to the path curves,

$$m\cdot n = 0, \qquad m\cdot m = \Sigma x_i'^2 = 1, \qquad n\cdot n = \Sigma x_i^2$$

and the element of arc then becomes

$$d\sigma^2 = ds^2 + \Sigma x_i^2 dt^2.$$

The expression Σx_i^2 is a function of s only. Hence *any of these surfaces can be mapped without stretching on a surface of revolution in ordinary space.* If (10) is a developable surface, we know from the theory of surfaces of revolution in ordinary space that

$$\Sigma x_i^2 = (as + b)^2. \tag{19}$$

There are surfaces besides cylinders and cones which satisfy this relation in space of four dimensions. In fact we do not have cones or cylinders of revolution for this kind of rotation. *The only ruled developable surfaces of revolution are planes.* For the only ruled developable surfaces are those formed by the tangents to a twisted curve.* If the surface is one of rotation, the edge of regression must be left invariant; but the only invariant curves are circles, consequently the only surfaces of revolution of this type are planes.

An interesting class of developables are those generated by curves lying in planes cutting two completely perpendicular planes in lines. For simplicity let us take a curve lying in the x_1x_3-plane. This is not a specialization since any such plane can be brought into this position by a simple change of coördinates. Using polar coördinates in this plane, we show that the condition for a developable surface becomes

$$r = as + b,$$

or

$$dr = a\sqrt{dr^2 + r^2 d\theta^2}.$$

The solution of this equation is

$$r = e^{\frac{a}{\sqrt{1-a^2}}(\theta - \theta_0)}.$$

Hence *logarithmic spirals lying in planes cutting two completely perpendicular planes in lines by this rotation generate developable surfaces.*

Formula (19) shows that any curve lying on a hypersphere with center at O will generate a developable surface. In particular a circle lying in the x_1x_3-plane and having center at O will generate a developable of revolution.

If a surface of revolution in a space of four dimensions can be mapped on a sphere in ordinary space, then

$$d\sigma^2 = a^2(d\theta^2 + \sin^2\theta d\varphi^2),$$

and if we make the transformation $\theta = s/a$, $\varphi = t$,

$$d\sigma^2 = ds^2 + a^2 \sin^2 \frac{s}{a} dt^2.$$

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* Wilson and Moore, loc. cit., p. 342.

This again represents a large class of surfaces but we will restrict the curve as above to lie in the x_1x_3-plane. Using polar coördinates again, we have

$$r = a \sin \frac{s}{a}, \qquad \text{or} \qquad s = a \sin^{-1} \frac{r}{a},$$

$$dr^2 + r^2 d\theta^2 = \frac{a^2 dr^2}{a^2 - r^2}.$$

The solution of this equation is

$$r = a \sin (\theta + \theta_0).$$

Hence, *circles passing through the origin and lying in a plane cutting a pair of completely perpendicular planes in lines will generate a surface of revolution which can be mapped on a sphere in a space of three dimensions.*

The surface formed by rotating a curve C lying in the x_1x_3-plane has two systems of plane sections. The planes of these two sets of curves form the two generations of a quadric hypercone with vertex at O. A plane of one system contains a perpendicular to each plane of the other system. If the plane of C passes through O but does not cut two completely perpendicular invariant planes, the planes of the two sets of plane sections still form the two generations of a quadric cone, but there is no perpendicularity relation between the planes. If the plane of C does not pass through O, one system of planes (those containing C) generate a quartic cone and the planes containing the path curves generate a cone with O as vertex, formed by projecting from O the ruled surface formed by the lines cutting C and a line in each of the planes (4). [These lines are obtained as the intersection of a space of three dimensions containing C and cutting the planes (4).] This ruled surface will, in general, be of order $2m$ where m is the order of C. Hence *the surface formed by rotating a plane curve of order m is, in general, of order $8m$.*

6. **Minimum surfaces.** The general condition that a surface be minimum is the vanishing of the vector mean curvature.* To express the vector mean curvature we need the three covariant derivatives and for this purpose I shall make use of the formulas developed by Wilson and Moore,†

$$(20) \qquad y_{11} = \frac{1}{a}\begin{vmatrix} m & n & p \\ m\cdot m & m\cdot n & m\cdot p \\ m\cdot n & n\cdot n & n\cdot p \end{vmatrix}, \qquad y_{12} = \frac{1}{a}\begin{vmatrix} m & n & q \\ m\cdot m & m\cdot n & m\cdot q \\ m\cdot n & n\cdot n & n\cdot q \end{vmatrix},$$

$$y_{22} = \frac{1}{a}\begin{vmatrix} m & n & r \\ m\cdot m & m\cdot n & m\cdot r \\ m\cdot n & n\cdot n & n\cdot r \end{vmatrix},$$

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* Wilson and Moore, loc. cit., p. 325.

† Loc. cit., pp. 337–8.

where $a = (m \cdot m)(n \cdot n) - (m \cdot n)^2$.

$$
(21) \quad \left\{ \begin{aligned}
m = \frac{\partial \rho}{\partial s} &= (x_1' \cos t - x_2' \sin t)k_1 + (x_1' \sin t + x_2' \cos t)k_2 \\
&\quad + (x_3' \cos t - x_4' \sin t)k_3 + (x_3' \sin t + x_4' \cos t)k_4, \\
n = \frac{\partial \rho}{\partial t} &= -\ (x_1 \sin t + x_2 \cos t)k_1 + (x_1 \cos t - x_2 \sin t)k_2 \\
&\quad - (x_3 \sin t + x_4 \cos t)k_3 + (x_3 \cos t - x_4 \sin t)k_4, \\
p = \frac{\partial^2 \rho}{\partial s^2} = \frac{\partial m}{\partial s} &= (x_1'' \cos t - x_2'' \sin t)k_1 + (x_1'' \sin t + x_2'' \cos t)k_2 \\
&\quad + (x_3'' \cos t - x_4'' \sin t)k_3 + (x_3'' \sin t + x_4'' \cos t)k_4, \\
q = \frac{\partial^2 \rho}{\partial s \partial t} = \frac{\partial m}{\partial t} &= -\ (x_1' \sin t + x_2' \cos t)k_1 + (x_1' \cos t - x_2' \sin t)k_2 \\
&\quad - (x_3' \sin t + x_4' \cos t)k_3 + (x_3' \cos t - x_4' \sin t)k_4, \\
r = \frac{\partial^2 \rho}{\partial t^2} = \frac{\partial n}{\partial t} &= -\ (x_1 \cos t - x_2 \sin t)k_1 - (x_1 \sin t + x_2 \cos t)k_2 \\
&\quad - (x_3 \cos t - x_4 \sin t)k_3 - (x_3 \sin t + x_4 \cos t)k_4.
\end{aligned} \right.
$$

Since s is the length of arc of the revolved curve

$$m \cdot m = \Sigma x_i'^2 = 1,$$

and by differentiating

$$\Sigma x_i' x_i'' = 0.$$

If we take the path curves perpendicular to the generating curve, that is take an orthogonal trajectory of the path curves for generating curve, we have also

$$m \cdot n = x_1' x_2 - x_2' x_1 + x_3' x_4 - x_4' x_3 = 0.$$

Differentiating this last relation, we also have

$$x_1'' x_2 - x_2'' x_1 + x_3'' x_4 - x_4'' x_3 = 0.$$

Using these relations, we find

$$
(22) \quad \left\{ \begin{array}{lll}
n \cdot n = \Sigma x_i^2, & m \cdot p = \Sigma x_i' x_i'' = 0, & m \cdot q = 0, \\
m \cdot r = -\ \Sigma x_i x_i', & n \cdot p = x_1 x_2'' - x_2 x_1'' + x_3 x_4'' - x_4 x_3'' = 0, & \\
n \cdot q = \Sigma x_i x_i', & n \cdot r = 0, \quad a = n \cdot n = \Sigma x_i^2. &
\end{array} \right.
$$

Substituting these values in (20), we have

$$(23) \qquad y_{11} = p, \qquad y_{12} = q - \frac{n \cdot q}{n \cdot n} n, \qquad y_{22} = r - (m \cdot r)m,$$

From (22) it is at once seen that y_{11}, y_{12}, y_{22} are normal to the surface which indeed is always true of the absolute derivatives. If the curvature vector of a curve, traced on a surface, lies in the normal plane to the surface the curve is a geodesic. Then from the first equation of (23) we see that *the orthogonal trajections of the path curves of a surface of revolution in space of four dimensions are geodesics.** The last equation of (23) shows that $m \cdot r = 0$ is a sufficient condition that the path curves be geodesics. The integral of this is

$$\Sigma x_i^2 = a^2.$$

The general condition that the path curves be geodesics is that r lie in the normal plane, that is that r be perpendicular to m and n. From (22) $n \cdot r = 0$, consequently $m \cdot r = 0$ is both necessary and sufficient. Hence *the only surfaces of revolution on which the path curves are geodesics are those formed by revolving a spherical curve.* From (19) we see that all such surfaces are developable. If $m \cdot r = 0$, then $n \cdot q = 0$, and (23) shows that p, q, r are three normals to the surface and consequently satisfy a linear relation.

The vector mean curvature is given by the formula (Wilson and Moore, p. 322)

$$2h = \Sigma a^{(rs)} y_{rs} = p + \frac{r}{n \cdot n} - \frac{m \cdot r}{n \cdot n} m.$$

The vanishing of this vector is the condition for a minimum surface. Then the coördinates of the curve will have to satisfy the differential equations

$$(24) \qquad (\sum_i x_i^2) x_j'' + (\sum_i x_i x_i') x_j' - x_j = 0, \qquad j = 1, 2, 3, 4.$$

Multiply the first of these equations by x_2 and the secondly $-x_1$ and add, then the first by x_3 and the third by $-x_1$ and add, and finally the first by x_4 and the fourth by $-x_1$ and add. We thus obtain

$$(25) \qquad \begin{cases} (\Sigma x_i^2)(x_1'' x_2 - x_2'' x_1) + \Sigma x_i x_i' (x_1' x_2 - x_2' x_1) = 0, \\ (\Sigma x_i^2)(x_1'' x_3 - x_3'' x_1) + \Sigma x_i x_i' (x_1' x_3 - x_3' x_1) = 0, \\ (\Sigma x_i^2)(x_1'' x_4 - x_4'' x_1) + \Sigma x_i x_i' (x_1' x_4 - x_4' x_1) = 0. \end{cases}$$

Eliminating the Σ's, we have

$$\frac{x_1'' x_2 - x_2'' x_1}{x_1' x_2 - x_2' x_1} = \frac{x_1'' x_3 - x_3'' x_1}{x_1' x_3 - x_3' x_1} = \frac{x_1'' x_4 - x_4'' x_1}{x_1' x_4 - x_4' x_1}.$$

The first integral is

$$x_1' x_2 - x_2' x_1 = C_1(x_1' x_3 - x_3' x_1) = C_2(x_1' x_4 - x_4' x_1).$$

The complete integral then is

$$\begin{aligned} x_1 &= A x_2 + B x_3, \\ x_2 &= C x_2 + D x_4. \end{aligned}$$

Hence if there exists a minimum surface it must be formed by rotating a plane curve but we saw that the only plane curves that are orthogonal to the path curves lie in planes cutting a pair of completely perpendicular invariant planes in lines. Then by a proper choice of axes we can take the curve in the x_1x_3-plane. Equations (24) then become

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* Cf. E. E. Levi, Sui gruppi di movimenti, Atti dei Lincei, Series 5, vol. XIV^1 (1905).

$$(x_1^2 + x_3^2)x_1'' + (x_1x_1' + x_3x_3')x_1' - x_1 = 0,$$
$$(x_1^2 + x_3^2)x_3'' + (x_1x_1' + x_3x_3')x_3' - x_3 = 0.$$

If we assume the relation

$$x_1'^2 + x_3'^2 = 1, \tag{26}$$

these equations are not independent, since then the sum of the first multiplied by x_1' and the second multiplied by x_3' is zero. On the assumption (26) the second equation reduces to

$$\frac{\dfrac{d^2x_3}{dx_1^2}}{1+\left(\dfrac{dx_3}{dx_1}\right)^2} + \frac{x_1\dfrac{dx_3}{dx_1} - x_3}{x_1^2 + x_3^2} = 0.$$

The first integral is

$$\frac{dx_3}{dx_1} = \frac{C_1x_1 - x_3}{x_1 + C_1x_3}$$

and the complete integral is

$$C_1(x_3^2 - x_1^2) + 2x_1x_3 + C_2 = 0.$$

Hence *equilateral hyperbolas with center at O and lying in planes which cut a pair of completely perpendicular invariant planes in lines generate minimum surfaces by this rotation.* These surfaces and planes are the only minimum surfaces left invariant by the rotation (2).

If $n \cdot n =$ const., equation (25) no longer holds but in this case the surface is developable and there are no minimum developable surfaces. For if the surface is minimum, the center of the indicatrix is at the surface point,* and if it is developable, the distance from the center of the indicatrix to the surface point is $\sqrt{a^2 + b^2}$, where a and b are the axes of the indicatrix. Therefore if the surface is both minimum and developable, the indicatrix must reduce to a point. The only real surface for which this is true is a plane.

7. **Surfaces for which the indicatrix reduces to a linear segment.** Another interesting class of surfaces in hyperspace are those for which the indicatrix reduces to a linear segment because those surfaces possess many of the properties of surfaces in three dimensions. The area of the indicatrix is given by the formula

$$a_{12}y_{22}xy_{11} + a_{22}y_{11}xy_{12} + a_{11}y_{12}xy_{22},$$

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* Wilson and Moore, loc. cit., p. 326.

where the cross indicates the Gibbs vector product and a_{rs} are the coefficients in the first fundamental form. Then

$$a_{11} = 1, \qquad a_{12} = 0, \qquad a_{22} = \Sigma x_i^2.$$

The vanishing of this invariant will then be the condition that the indicatrix reduce to a linear segment. Substituting the values of the quantities previously found this reduces to

$$(27) \qquad [(\Sigma x_i^2)p - r + (m\cdot r)m] \times [q - (n\cdot q)n] = 0.$$

This product will vanish if either factor vanishes or if the two vectors are parallel. If the first factor vanishes, we have

$$(28) \qquad (\Sigma x_i^2)x_j'' - (\Sigma x_i x_i')x_j' + x_j = 0, \qquad j = 1, 2, 3, 4.$$

Multiply by x_j and sum on j; we obtain

$$(\sum_i x_i^2)(\sum_j x_j x_j'' + 1) - (\sum_i x_i x_i')^2 = 0.$$

The complete integral is

$$\Sigma x_i^2 = (C_1 s + C_2)^2.$$

Hence the only surfaces for which the first factor of (27) vanishes are developables. The set of equations (28) when treated the same as we did (24) lead to the same set of equations showing that the surfaces must be generated by rotating plane curves whose plane cuts a pair of completely perpendicular invariant planes. If $m\cdot n$ = const., the argument does not apply. In this case the surface of rotation lies on a sphere and (27) becomes

$$(C^2 p - r) \times q = 0,$$

and if the first factor of this vanishes

$$C^2 x_j'' + x_j = 0, \qquad j = 1, 2, 3, 4.$$

The solution is

$$x_j = A_j \cos\frac{s}{c} + B_j \sin\frac{s}{c}.$$

This is a plane curve and therefore a circle and can be taken in the x_1x_3-plane.

If the second factor of (27) vanishes,

$$(\sum_i x_i^2)x_j' - (\Sigma x_i x_i')x_j = 0, \qquad j = 1, 2, 3, 4.$$

The solution is seen to be a straight line which by the rotation (2) forms a plane.

Finally if the factors are parallel vectors, we have

$$\frac{(\sum_i x_i^2)x_j'' - (\sum_i x_i x_i')x_j' + x_j}{(\Sigma x_i^2)x_j' - (\Sigma x_i x_i')x_j} = \text{const.}, \qquad j = 1, 2, 3, 4.$$

Multiplying the numerators and denominators respectively by x_1', x_2', x_3', x_4' and adding, we find the new numerator vanishes and consequently the denominators must vanish and we have

$$\Sigma x_i^2 - (\Sigma x_i x_i')^2 = 0.$$

The integral is

$$\Sigma x_i^2 = (s + c)^2.$$

Since Σx_i^2 is the distance from the origin to a point of the curve it is obvious that the only curves that will satisfy this relation are lines passing through the origin. *Besides planes the only surfaces of revolution for which the indicatrix reduces to a linear segment are developables formed by rotating plane curves.*

8. **Locus of the indicatrix.** The indicatrix is an ellipse lying in the normal plane and if the normal plane is revolved, in each position it will contain the indicatrix at the new point. Hence the locus of the indicatrix corresponding to points of a path curve on a surface of revolution is a surface of order 16 having one set of equal conics and one set of circles with center at O for plane sections. If the indicatrix reduces to a straight line the locus becomes a ruled surface of revolution and consequently of order 8. The end of the mean curvature vector will generate a surface of revolution on which the circles (path curves) will correspond to the circles on the original surface of revolution. The circles on this surface are the locus of the center of the indicatrix corresponding to circles of the original surface.

9. **The General rotation.** If instead of the special rotation (2) we use the general rotation (1) having the origin for fixed point, we obtain a surface of revolution having many of the characteristics of this special one. In this case however the path curves are not circles and indeed if m_1 and m_2 are incommensurable the path curves are not even closed curves. If we consider $m_1 : m_2$ and t as parameters we obtain the surface left invariant by the two parameter group having the same two fixed planes. This is a developable surface traced on the sphere.* The cartesian equations of this surface are

$$x_1^2 + x_2^2 = a^2, \qquad x_3^2 + x_4^2 = b^2.$$

Many of the properties of this surface are given in my article just referred to.

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* Cf. C. L. E. Moore, "Rotations in hyperspace," Proceedings of the American Academy of Arts and Sciences, vol. 53 (1918), p. 676.

For this type of rotation there are ruled developable surfaces of revolution, viz., the surface formed by revolving the tangent lines to a path curve. The locus of a straight line or a plane are in this case transcendental.

For this general rotation surfaces traced on a sphere are not necessarily developable. The general developable satisfies the condition

$$m_1^2(x_1^2 + x_2^2) + m_2^2(x_3^2 + x_4^2) = (as + b)^2,$$

from which we see that surfaces traced on a sphere for the special rotation correspond here to the surfaces traced on the ellipsoid

$$m_1^2(x_1^2 + x_2^2) + m_2^2(x_3^2 + x_4^2) = b^2.$$

Two pairs of axes of this ellipsoid are equal.

The condition that the curve rotated be an orthogonal trajectory of the path curves is

$$m_1(x_1'x_2 - x_2'x_1) + m_2(x_3'x_4 - x_4'x_3) = 0.$$

Equations (23) remain the same from which we see that *the orthogonal trajectories of the path curves are geodesics.* Also the only surfaces for which the path curves are geodesics are the surfaces traced on the ellipsoid

$$m_1^2(x_1^2 + x_2^2) + m_2^2(x_3^2 + x_4^2) = b^2.$$

It is easily seen also that the indicatrix of surfaces traced on this ellipsoid always degenerates into a linear segment.

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